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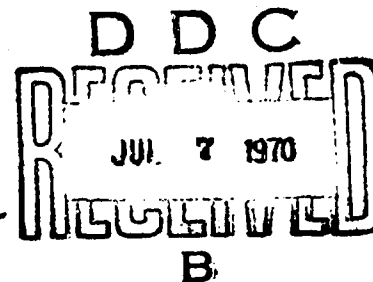
NRL Report 5267

THEORY OF A VERY THIN PIEZOCERAMIC HOLLOW SPHERE UNDERWATER SOUND RADIATOR

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March 6, 1959



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Washington, D.C.

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ABSTRACT

The theory of the thin shell hollow sphere piezoceramic underwater sound radiator is investigated in detail. Starting with the equations of state an equation of motion is derived for the case of pure radial motion. Formulas are deduced for coefficient of electromechanical coupling, velocity, power conversion, mechanical Q, transmitting pressure, impedance, receiving response, and efficiency. Design charts based on the properties of two popular ceramics (a BaTiO_3 - CaTiO_3 mix and a PbTiO_3 - PbZrO_3 mix) are also presented. The limits of validity of the derived equation are also discussed.

PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing.

AUTHORIZATION

NRL Problem S02-05
Projects NR 441-000, Task NR 441-004,
and NE 051-600.26
BuShips No. S-1692

Manuscript submitted December 19, 1958

THEORY OF A VERY THIN PIEZOCERAMIC HOLLOW SPHERE UNDERWATER SOUND RADIATOR

INTRODUCTION

The vibrations of an isotropic elastic sphere, or shell bounded by two spherical surfaces, have been worked out in detail by various writers. Poisson (1) in his memoir of 1829 discussed the radial vibration of a solid sphere. Jaerisch (2), Chree (3), and Lamb (4) investigated problems in the free and forced vibrations of solid and hollow spherical shells. Bromwich (5), Jeans (6), and Love (7) extended these investigations to include geophysical phenomena. Laura (8) worked out expressions for the effect of a liquid medium on a vibrating sphere. In more recent times the problem of the acoustic vibrations of a polarized electrostrictive hollow sphere in a liquid medium under electrical excitation has been studied by Rosenthal and Baerwald (9).

The following analysis recapitulates the work of Rosenthal and Baerwald and proceeds in greater detail to derive results not found in their report. The usefulness of many of the derived equations is improved by presentation of design graphs in Appendix A.

THE STRUCTURE

The structure (Fig. 1) consists of a hollow sphere of polarized electrostrictive ceramic made of two hemispheres cemented at the equator. Each surface of the sphere is completely electroded with fired silver paste. Polarization of the ceramic is in the direction along a radial line. The structure is driven into forced vibration by the application of an alternating electric field (E_3) applied across the thickness of the sphere.

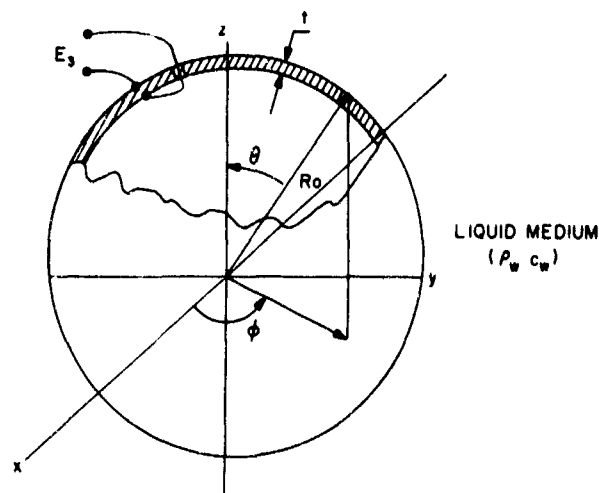


Figure 1 - Piezoceramic hollow sphere
underwater sound radiator

The vibrating sphere radiates sound into an infinite medium of density ρ_w and sound velocity (compressional waves) c_w .

EQUATIONS OF STATE

The analysis will be conducted in the orthogonal spherical system of coordinates, θ (colatitude angle), ϕ (azimuth angle), and r (radius). For simplification in writing, when these appear as subscripts they will be designated by the numerals 1, 2, 3 respectively. It is assumed from the onset that the thickness-to-exterior radius ratio t/R_0 is considerably less than unity, and that the variation of mechanical fields (stress) and electrical fields (electric field gradient) with the radial coordinate is negligible. The stress field and electric field are thus two dimensional, the planar character of which will be emphasized by placing a bar over all symbols representing the material constants (stiffness constant \bar{c}_{11} , piezo modulus \bar{e}_{31} , etc.).*

The appropriate set of equations of state should begin with the physical fact that electromechanical conversion of energy is obtained by exciting a polarized electrostrictive material. However, it is more convenient to treat the active ceramic as an artificial piezoelectric crystal and assign to it certain "effective moduli" whose form and magnitude are determinable by conventional piezoelectric techniques. In accordance with this procedure we describe the planar electromechanical state of the polarized ceramic by a set of linear piezoelectric equations in the form suggested by Mason (10). These equations state that the planar stresses (T_1, T_2) depend upon the state of strain (S_1, S_2) and upon the electric field (E_3) in an infinitesimal volume of material anywhere inside the ceramic in the following way:

$$\left. \begin{aligned} T_1 &= \bar{c}_{11}^E S_1 + \bar{c}_{12}^E S_2 - \bar{e}_{31} E_3 \\ T_2 &= \bar{c}_{12}^E S_1 + \bar{c}_{11}^E S_2 - \bar{e}_{31} E_3 \end{aligned} \right\} \quad (1a)$$

The dependence of the electric displacement vector D_3 on the electric field and the state of strain is

$$D_3 = \epsilon_{33}^S E_3 + \bar{e}_{31}(S_1 + S_2) \quad (1b)$$

where ϵ_{33}^S is the dielectric constant at zero strain. A positive strain is defined as one that accompanies a state of mechanical tension in a body free to move. A positive electric field vector is one that points in the same direction as the polarizing electric field vector.

The mechanical equations (1a) and the electrical equation (1b) are independent statements of laws of nature having in each case the same two independent variables E and S . If one of these variables (say S) is eliminated by substitution, a single equation of state is obtained which describes conditions at any material point of the ceramic due to both electrical and mechanical fields. Equation (1b) becomes

$$D_3 = \epsilon_{33}^S \left[1 + \frac{2\bar{e}_{31}^2}{(\bar{c}_{11}^E + \bar{c}_{12}^E)\epsilon_{33}^S} \right] E_3 + \frac{\bar{e}_{31}(T_1 + T_2)}{\bar{c}_{11}^E + \bar{c}_{12}^E} \quad (2a)$$

In a similar way the variable E_3 may be eliminated between the equations, and Eqs. (1a) become

*A list of symbols appears at the end of the report.

$$T_1 + T_2 = (\bar{\epsilon}_{11}^E + \bar{\epsilon}_{12}^E) \left[1 + \frac{2\bar{\epsilon}_{31}^2}{(\bar{\epsilon}_{11}^E + \bar{\epsilon}_{12}^E)\epsilon_{33}^S} \right] (S_1 + S_2) - \frac{2\bar{\epsilon}_{31} D_3}{\epsilon_{33}^S}. \quad (2b)$$

It is noted in each of Eqs. (2a) and (2b) that the magnitude of the material constants ϵ_{33}^S , $\bar{\epsilon}_{11}^E$, and $\bar{\epsilon}_{12}^E$ is modified by the same group of assembled symbols. This group is defined in the literature (11) as the material static mixed planar coefficient of electro-mechanical coupling, $(k_p^2)_{mix}$; that is,

$$(k_p^2)_{mix} = \frac{2\bar{\epsilon}_{31}^2}{(\bar{\epsilon}_{11}^E + \bar{\epsilon}_{12}^E)\epsilon_{33}^S}. \quad (3a)$$

A related parameter of electromechanical coupling that will prove of use in later discussions is $(k_p^2)_{hom}$, where

$$(k_p^2)_{hom} = \frac{(k_p^2)_{mix}}{1 + (k_p^2)_{mix}}. \quad (3b)$$

For convenience in writing we can reduce the number of terms appearing in Eq. (2b) by recalling that $\bar{\epsilon}_{11}^E/\bar{\epsilon}_{12}^E = \nu$ (Poisson's ratio) and by defining a new constant $\bar{\epsilon}_{11}^D$ such that

$$\bar{\epsilon}_{11}^D = \bar{\epsilon}_{11}^E [1 + (k_p^2)_{mix}].$$

The tangential strain S_1 ($S_1 = S_2$) for a purely radial symmetrical mode of motion, is simply the radial displacement ξ divided by the external radius R_0 . Equation (2b) may therefore be written

$$T_1 + T_2 = 2\bar{\epsilon}_{11}^D (1 + \nu) \frac{\xi}{R_0} - \frac{2\bar{\epsilon}_{31} D_3}{\epsilon_{33}^S}. \quad (4)$$

In this equation the coupling of the electrical field to the mechanical is implicitly contained in the open circuit stiffness constant $\bar{\epsilon}_{11}^D$. While it is always possible to use Eqs. (1a) in the form shown as an aid in determining the radial displacement ξ , the displacement so determined would be a first approximation only, since Eq. (1b) would still remain to be satisfied. The method of initial substitution used here is convenient, though superficial, and has the disadvantage of changing the field variable E_3 to the electric displacement variable D_3 . However, the dynamic equation of motion is directly deducible from Eq. (4) without the preliminary inversion of factors that accompanies the use of equations involving the piezo modulus d_{31} . From this point of view the method leading to Eq. (4) is preferable to other possible methods, though the end results of all correct methods will differ one from the other in no material way.

EQUATION OF MOTION AND SOLUTIONS

The stress equation of dynamic equilibrium for the case of pure radial motion in spherical coordinates is (12)

$$\frac{\partial T_{33}}{\partial r} + \frac{1}{r} \frac{\partial T_{31}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{32}}{\partial \phi} + \frac{1}{r} [2T_{33} - T_{11} - T_{22} + T_{31} \cot \theta] + \rho_c F_r = \rho_c \frac{d^2 \xi}{dt^2}. \quad (5)$$

All the terms appearing in this relation are forces acting upon a unit of (interior) volume of material with density ρ_c . In particular, the quantity F_r is the body force per unit of

mass arising from some external agency. This differs in its nature from the elastic forces accompanying deformation, which are described by the stress derivatives above.

Under the assumptions of thin shell planar analysis (restricted to pure radial motion) the forces stemming from the stresses T_{33} , T_{31} , and T_{32} and their derivatives are to be neglected. In addition the external agency of (acoustic) fluid pressure p on the shell of thickness t is considered to induce a body force per unit of volume equal in magnitude to p/t . The direction of this "body force" must be carefully chosen to agree with the physical situation of a pulsating shell. When the planar stresses T_{11} and T_{22} are positive (tension), the radial displacement ξ is positive, since the shell is expanding. Under the action of elastic restoring forces the acceleration $d^2\xi/dt^2$ must however be negative, that is, toward the origin of coordinates. Simultaneously the external acoustic pressure is negative (compression), and induces a negative acceleration. The equation of motion therefore for a very thin spherical shell of external radius R_0 reduces to

$$\frac{-(T_{11} + T_{22})}{R_0} + \frac{p}{t} = \rho_c \frac{d^2\xi}{dt^2}. \quad (6)$$

The time variation of ξ has not been specified, and while pulsed excitation of the sphere would prove of interest, it is the harmonic response that is of principal utility. In addition, since we have made acoustic pressure, p , positive when it applies to deficit pressure (tension), contrary to the conventional formulas of acoustic theory which make excess pressure positive, we modify the latter accordingly and write the known specific acoustic impedance of a pulsating sphere with an additional negative sign; that is, we write (13)

$$p = \frac{-\rho_w c_w k r (kr + j)}{1 + k^2 r^2} v, \quad (7)$$

where k is the wave number, and v is the medium velocity here identified with $j\omega\xi$.

Substituting Eqs. (4) and (7) into (6) and collecting terms, we obtain

$$-\omega^2 \xi \rho_c + \frac{\rho_w c_w k R_0 (k R_0 + j)}{t(1 + k^2 R_0^2)} j\omega \xi + 2\bar{\epsilon}_{11}^D (1 + \nu) \frac{\xi_0}{R_0^2} = \frac{2\bar{\epsilon}_{31} D_3}{\epsilon_{33}^S R_0}. \quad (8)$$

Equation (8) is a differential relation expressing the forced excitation of the sphere by an electromechanical force per unit volume F_0 equal to the quantity $2\bar{\epsilon}_{31} D_3 / \epsilon_{33}^S R_0$. The radial mechanical displacement is therefore

$$\xi = \frac{\bar{\epsilon}_{31} R_0 D_3 e^{j\omega t}}{2\bar{\epsilon}_{11}^D (1 + \nu) \epsilon_{33}^S \left\{ 1 - \left(\frac{\omega}{\omega_n} \right)^2 - \frac{\rho_w}{\rho_c} \frac{R_0}{t} \left(\frac{\omega}{\omega_n} \right)^2 \left[\frac{1 - \frac{j\omega R_0}{c_w}}{1 + \left(\frac{\omega R_0}{c_w} \right)^2} \right] \right\}} \quad (9)$$

where

$$\omega_n^2 = \frac{2\bar{\epsilon}_{11}^D (1 + \nu)}{R_0^2 \rho_c}. \quad (10)$$

The quantity ω_n defined by Eq. (10) has a special significance. In the absence of an acoustic load (i.e., when $\rho_w = 0$) ω_n is that frequency which produces an infinite displacement, internal dissipation neglected. It is therefore the ideal mechanical resonant frequency of a radially pulsating thin shell sphere under open circuit conditions.

A noteworthy feature of Eq. (9) is that the displacement is a complex number, resulting from the existence of velocity-dependent and acceleration-dependent loads. The absolute magnitude of this complex displacement is found to be

$$|\xi| = (\xi \xi^*)^{1/2} = \frac{\bar{\epsilon}_{31} R_o D_3}{\bar{\epsilon}_{11}^D (1 + \nu) \epsilon_{33}^S \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 2 \left(\frac{\omega}{\omega_n}\right)^2 \frac{\rho_w}{\rho_c} \frac{R_o}{t \left[1 + \left(\frac{\omega R_o}{c_w}\right)^2\right]} \left[\left(\frac{\omega}{\omega_n}\right)^2 \left(\frac{1}{2} \frac{\rho_w R_o}{\rho_c t} + 1\right) - 1\right]}} \quad (11)$$

Further deductions from Eq. (8) may be obtained by writing it in a more revealing way. Again restricting the time dependence to steady state, and using the sinusoidal forcing excitation F_o , we find that

$$-m^* \omega^2 \xi_o + r^* j \omega \xi_o + k^* \xi_o = F_o \quad (12a)$$

where

$$m^* = \rho_c \left[1 + \frac{\rho_w}{\rho_c} \frac{R_o}{t(1 + k^2 R_o^2)} \right] \quad (12b)$$

$$r^* = \frac{\rho_w c_w k^2 R_o^2}{t(1 + k^2 R_o^2)} \quad (12c)$$

$$k^* = \frac{2 \bar{\epsilon}_{11}^D (1 + \nu)}{R_o^2} \quad (12d)$$

Equation (12a) is recognized as the steady state differential equation of forced motion of a velocity-damped mass-spring system, with frequency-dependent constants. The solution is easily seen to be

$$\xi = \frac{F_o \cos(\omega t - \phi)}{\sqrt{r^{*2} \omega^2 + (\omega^2 m^* - k^*)^2}} \quad (13a)$$

$$\tan \phi = \frac{r^* \omega}{-\omega^2 m^* + k^*} \quad (13b)$$

The mechanical resonant frequency in the liquid medium ω_R is defined as that frequency at which the phase angle ϕ between F_o and ξ is $\pi/2$. That is,

$$\omega_R^2 = \frac{k^*}{m^*}$$

or

$$\omega_R^2 = \frac{2 \bar{\epsilon}_{11}^D (1 + \nu)}{R_o^2 \rho_c \left\{ 1 + \frac{\rho_w}{\rho_c} \frac{R_o}{t \left[1 + \left(\frac{\omega_R R_o}{c_w} \right)^2 \right]} \right\}} \quad (14)$$

The difference between the mechanical resonant frequency in air, Eq. (10), and the mechanical resonant frequency in a liquid, Eq. (14), thus depends upon the parameter in

brackets. This latter quantity, which may be thought of as a factor by which the density of the ceramic is to be increased, is itself frequency dependent, and hence not determinate, unless the magnitude of ω_R is already known. Equation (14) is therefore best solved by trial and error. However, in any given situation it will be convenient to assume $\omega_R \approx \omega_n$ in order to compute this density modifying factor. Repeated trials will quickly yield the true value of ω_R .

Further deductions from Eq. (13a) may be made by proceeding conventionally to define the quality factor Q_m^* and the resonance factor n in such a way that we may write

$$\xi = \frac{F_0 \cos(\omega t - \phi)}{k^* \sqrt{(1 - n^2)^2 + \frac{n^2}{Q_m^{*2}}}} \quad (15a)$$

$$n = \omega/\omega_R \quad (15b)$$

$$Q_m^* = \omega_R m^* / r^* \quad (15c)$$

The mechanical Q_m is defined as the quality factor Q_m^* computed at velocity resonance, i.e., at the frequency for which the phase angle is $-\pi/2$. If we designate the wave parameter $k(=\omega/c_w)$ corresponding to the frequency of velocity resonance as k_R , we derive an expression for Q_m when $\omega = \omega_R$:

$$Q_m = \frac{k_R R_0 \left[1 + \frac{\rho_w}{\rho_c} \frac{R_0}{t(1 + k_R^2 R_0^2)} \right]}{\frac{\rho_w}{\rho_c} \frac{R_0 k_R^2 R_0^2}{t(1 + k_R^2 R_0^2)}} \quad (16a)$$

In general, $k_R^2 R_0^2 \gg 1$ for the case of a hollow thin shell at velocity resonance. The mechanical Q_m thus reduces to

$$Q_m \approx \frac{(k_R R_0) \rho_c t}{\rho_w R_0} + \frac{1}{k_R R_0} \quad (16b)$$

Expressions for surface velocity and surface acceleration are

$$\dot{\xi} = \frac{-F_0 \sin(\omega t - \phi)}{\sqrt{r^{*2} + \left(\omega m^* - \frac{k^*}{\omega}\right)^2}} \quad (17a)$$

$$\ddot{\xi} = \frac{-\omega F_0 \cos(\omega t - \phi)}{\sqrt{r^{*2} + \left(\omega m^* - \frac{k^*}{\omega}\right)^2}} \quad (17b)$$

By suitable maximizing procedures the frequencies of maximum displacement (ω_d) and maximum acceleration (ω_a) are derived to be

$$\omega_d = \omega_R \sqrt{1 - \frac{1}{2Q_m^2}} \quad (18a)$$

$$\omega_a = \omega_R \sqrt{1 - \frac{1}{2Q_m^2}} \quad (18b)$$

Because of the simplicity of the equation of motion, Eq. (12), the derived expressions, Eqs. (15), (17), and (18), were quickly obtained, since they are standard results. An additional standard result, namely, the temporal damping factor α_d , may be obtained with similar ease, since

$$\alpha_d = \frac{r^*}{2m^*}$$

and thus

$$\alpha_d = \frac{1}{2} \frac{\rho_w c_w}{\rho_c t} \left[\frac{k_R^2 R_o^2}{1 + k_R^2 R_o^2 + \frac{\rho_w R_o}{\rho_c t}} \right] \quad (19)$$

The reduction of the problem of the radially pulsating sphere to that of a system with a single degree of freedom is a convenient, though not wholly satisfactory, approximation. Since the stiffness per unit volume is k^* , the total stiffness is evidently $4\pi R_o^2 t k^*$, a quantity proportional to the thickness t . Having idealized the elastic system to the extent of allowing t^2/R_o^2 to be very small, it is seen that in the limit the hollow sphere will have no stiffness at all ($t \approx 0$). It will resemble a membrane, which, for any given surface force, yields extremely large displacements. The acoustic performance of such an idealized structure will be extraordinary. Due caution is therefore to be observed in the intended uses of the following performance formulas.

ELECTRICAL IMPEDANCE

The solution of the equation of motion has led to an expression for mechanical displacement ξ in terms of electric displacement D_3 (Eq. 9). Substituting this into Eq. (1b) and solving for E_3 we obtain

$$E_3 = \frac{D_3}{\epsilon_{33}} \left\{ 1 - \frac{(k_p^2)_{\text{eff}}}{\left[1 - \left(\frac{\omega}{\omega_h} \right)^2 - \frac{\rho_w R_o}{\rho_c t} \left(\frac{\omega}{\omega_h} \right)^2 \frac{\left(1 - j \frac{\omega R_o}{c_w} \right)}{1 + \left(\frac{\omega R_o}{c_w} \right)^2} \right]} \right\} e^{j\omega t} \quad (20)$$

For the case of a very thin shell, the current I_3 is $j\omega 4\pi R_o^2 D_3$, and the dependent voltage V_3 is approximately $E_3 t$. Hence we may rewrite Eq. (20) as a typical impedance relation:

$$\frac{V_3}{I_3} = \frac{1}{j\omega C^*} \left\{ 1 - \frac{(k_p^2)_{\text{eff}}}{\left[1 - \left(\frac{\omega}{\omega_h} \right)^2 - \frac{\rho_w R_o}{\rho_c t} \left(\frac{\omega}{\omega_h} \right)^2 \frac{\left(1 - j \frac{\omega R_o}{c_w} \right)}{1 + \left(\frac{\omega R_o}{c_w} \right)^2} \right]} \right\} \quad (21)$$

where

$$C^* = 4\pi R_o^2 \epsilon_{33} / t$$

A numerical computation using Eq. (21) will prove instructive. We assume a hollow shell (in water) of lead titanate-lead zirconate ceramic for which $(k_p^2)_{\text{eff}} = 0.25$, $\epsilon_{33} = \epsilon_o$

at air resonance is 4.0, $R_0/t = 10$, and $\rho_w/\rho_c = 1/7.6$. If the drive frequency is the same as the mechanical resonant frequency in air (if $\omega = \omega_n$) then

$$\frac{I_3}{V_3} \approx 0.596 j\omega C^* + 0.381 \omega C^* = jB_1 + G_1.$$

As seen from the electrical terminals the sphere appears as a capacity in parallel with a resistance. The electrical $Q (= B_1/G_1)$ is approximately 1.5. Due to the inertial effect of the water the susceptance jB_1 is reduced to 0.6 of its low frequency value, a remarkable reduction, stemming from the high coupling factor.

ACOUSTIC POWER DELIVERED

From Eq. (12) it is apparent that the force per unit volume dissipated in the resistance r^* is ξr^* . Since every particle of the sphere moves with the same displacement, the total force in the direction of the velocity ξ is $4\pi R_0^2 t \xi r^*$. Recalling that D_3 is an independent variable (constant current source) we employ Eq. (17a) and write an expression for the maximum acoustic power delivered to the medium:

$$(P_a)_{max} = (4\pi R_0^2 t) r^* \xi \dot{\xi}$$

or

$$P_a = \frac{16\pi t r^* \bar{\epsilon}_{31}^2 D_3^2 \sin^2(\omega t - \phi)}{\epsilon_{33}^S \left[r^{*2} + \left(\omega m^* - \frac{k^*}{\omega} \right)^2 \right]}. \quad (22)$$

Since the applied current I_3 is equal to $4\pi R_0^2 j\omega D_3$, and the blocked capacity C^* is $4\pi R_0^2 \epsilon_{33}^S/t$, Eq. (22) can be rewritten in terms of I_3 :

$$P_a = \frac{16\pi}{t} \left(\frac{I_3}{\omega C^*} \right)^2 \frac{\bar{\epsilon}_{31}^2 r^* \sin^2(\omega t - \phi)}{\left[r^{*2} + \left(\omega m^* - \frac{k^*}{\omega} \right)^2 \right]}. \quad (23)$$

It is frequently desirable to have an expression for the acoustic power in terms of the applied (or source) voltage V_3 . Regarding the source current I_3 as a constant source in parallel with its admittance $j\omega C^*$, both supplying the radiation resistance r^* with power, we see that the equivalent voltage source is V_3 in series with the impedance $1/j\omega C^*$, both supplying the same resistance r^* . Hence $|I_3/\omega C^*|$ is equal to V_3 , the peak applied voltage.

Employing Eq. (3b) we can write Eq. (23) in terms of the coupling factor $(k_p^2)_{hom}$:

$$P_a = \frac{8\pi (k_p^2)_{hom} \bar{\epsilon}_{11}^E (1+\nu) [1 + (k_p^2)_{mix}] \epsilon_{33}^S r^* V_3^2 \sin^2(\omega t - \phi)}{t \left[r^{*2} + \left(\omega m^* - \frac{k^*}{\omega} \right)^2 \right]}. \quad (24)$$

At the frequency of velocity resonance ($\phi = 0$) the rms acoustic power into the water is therefore

$$(P_a)_{\omega=\omega_R} = \frac{4\pi (k_p^2)_{hom} \bar{\epsilon}_{11}^E (1+\nu) [1 + (k_p^2)_{mix}] \epsilon_{33}^S (1 + k_R^2 R_0^2) V_3^2}{\rho_w c_w k_R^2 R_0^2}. \quad (25)$$

It is to be remembered that V_3 in Eqs. (24) and (25) is the peak voltage.

Equation (25) shows that the acoustic power output at velocity resonance is approximately independent of the radius R_o . To reveal this fact we note that

$$k_R^2 R_o^2 = \left(\frac{\omega R_o}{c_w} \right)^2 = \left(\frac{\omega}{\omega_n} \right)^2 \left[\frac{2 \bar{c}_{11}^E (1 + \nu)}{\rho_c c_w^2} \right].$$

When $\omega = \omega_n$, we find that

$$k_R^2 R_o^2 = 2 \bar{c}_{11}^E (1 + \nu) / \rho_c c_w^2.$$

which is a factor independent of R_o . When $R \ll R_n$ this conclusion fails.

While the total resonant power per voltage squared is approximately independent of R_o , the surface power density (watts/cm²) is not. Hence a large resonant sphere (a low-frequency sphere) can radiate much more energy than a small resonant sphere, because it can tolerate a larger voltage (that is, for equal maximum surface power density, its voltage drive will be very much larger).

An illustrative computation using Eq. (25) will prove of interest. We select a lead titanate-lead zirconate material for which $(k_p^2)_{hom} = 0.23$, $\bar{c}_{11}^E = 91.3 \times 10^9$, $\nu = 0.3$, $1 + (k_p^2)_{mis} \approx 1 + (k_p^2)_{hom}$, $\epsilon_{33}^S = 7.89 \times 10^{-9}$, and $\rho_c = 7.6 \times 10^3$. At velocity resonance

$$k_R^2 R_o^2 = \frac{2 \times 91.3 \times 10^9 (1.3)}{7.6 \times 10^3 \times 1.55^2 \times 10^6} = 13.0.$$

Hence

$$P_s = \frac{4\pi \times 0.23 \times 91.3 \times 10^9 (1.3) (1.23) 7.89 \times 10^{-9} \times 14}{1.55 \times 10^6 \times 13} V^2$$

$$= 2310 \times 10^{-6} V^2 \text{ (rms power: } V = \text{peak volts).}$$

For a sphere 5 in. in diameter and 1/4 in. thick driven with 850 peak volts, the rms power would be 1670 watts at 100-percent electromechanical efficiency.

We may also deduce a formula for the voltage required to achieve a surface power density δ (watts/m²) based upon the restricted case of Eq. (25). At velocity resonance, we find

$$V_3 = \frac{\sqrt{2\delta} R_o}{(k_p)_{hom} \sqrt{1 + (k_p^2)_{mis}} \frac{\rho_c}{\rho_w} \epsilon_{33}^S c_w (1 + k_R^2 R_o^2)} \quad (26)$$

Example: To achieve a density of 3 watts/cm² using the 5-in. sphere of the material noted above we require

$$V_3 = \frac{\sqrt{2 \times 3 \times 10^4 \times 6.35 \times 10^{-2}}}{0.48 \sqrt{(1.23)(7.6) 7.89 \times 10^{-9} \times 1.5 \times 10^3 (14)}} = 825 \text{ peak volts.}$$

ACOUSTIC PRESSURE IN THE FAR FIELD

The expression for acoustic power obtained in the previous section may be written symbolically as the product of the particle pressure and the conjugate particle volume velocity. If the particle velocity is u , the latter quantity is $4\pi r^2 u^*$, since all parts of the

sphere vibrate in phase with each other. Now in the far field at distance R the particle pressure and velocity are substantially in phase and are related to one another through the characteristic impedance of the medium $\rho_w c_w$. The conjugate volume velocity is then $4\pi R^2 p^* / \rho_w c_w$. The magnitude of acoustic power in the far field therefore becomes

$$|P_a| = \frac{4\pi R^2}{\rho_w c_w} |p|^2.$$

Equating this magnitude of acoustic power with the magnitude in Eq. (24), and solving for the peak acoustic pressure in the far field, we obtain

$$|p| = \frac{\sqrt{2} (k_p)_{hom} V_3}{R} \sqrt{\frac{\rho_w c_w \bar{c}_{11}^E (1+\nu) [1 + (k_p^2)_{mix}] \epsilon_{33}^S r^*}{t \left[r^{*2} + \left(\omega m^* - \frac{k^*}{\omega} \right)^2 \right]}} \quad (27)$$

At velocity resonance, $\omega m^* = k^* / \omega$, and, after substituting Eq. (12c), the far field magnitude of acoustic pressure reduces to

$$|p| = \frac{\sqrt{2} (k_p)_{hom} V_3}{R} \frac{\sqrt{\bar{c}_{11}^E (1+\nu) [1 + (k_p^2)_{mix}] \epsilon_{33}^S (1 + k_R^2 R_o^2)}}{k_R R_o} \quad (28a)$$

$$|p| = \frac{\sqrt{2} (k_p)_{hom} I_3}{R \omega C^a} \frac{\sqrt{\bar{c}_{11}^E (1+\nu) [1 + (k_p^2)_{mix}] \epsilon_{33}^S (1 + k_R^2 R_o^2)}}{k_R R_o} \quad (28b)$$

At velocity resonance, the magnitude of the far field pressure is approximately independent of the radius of the sphere, provided the frequency in the liquid medium does not differ appreciably from the frequency in air. This will be the case for radius-to-thickness ratios (R_o/t) less than 20. A sample calculation using the constants of the 5-in. lead titanate-lead zirconate sphere mentioned above (in which $R_o/t = 10$) shows the value of the quantity $|p|/V_3$ to be

$$\begin{aligned} \frac{|p|}{V_3} &= \frac{\sqrt{2} \times 0.48 \sqrt{91.3 \times 10^9 \times 1.3 \times 1.23 \times 7.89 \times 10^{-9} \times 14}}{R \sqrt{13}} \\ &= \frac{23.8}{R} \frac{\text{newton}}{\text{meter-volt}} = \frac{47.6 \text{ db ref. } 1 \mu\text{bar at 1 meter}}{\text{volt}} \end{aligned}$$

Also, since

$$\begin{aligned} \omega C^a &= 2\pi \times 14 \times 10^3 \times 57,916 \times 10^{-12} \\ &= 465 \times 10^{-5} \end{aligned}$$

we have

$$\frac{|p|}{I} = \frac{5100}{R} \frac{\text{newton}}{\text{meter-ampere}} = \frac{94.2 \text{ db ref. } 1 \mu\text{bar at 1 meter}}{\text{ampere}}.$$

These are the maximum values to be expected since the efficiency of electromechanical conversion is assumed to be 100 percent.

RECEIVING RESPONSE

As long as the square of the acoustic pressure in the far field is strictly proportional to applied electrical power (linear operation) the acoustic performance of the sphere will obey the reciprocity law of linear transducers. Defining the reciprocity factor J for distance R and wavelength λ as

$$J = \frac{2R\lambda}{\rho_w c_w} \quad \left(J' = \frac{2\lambda}{\rho_w c_w} \text{ for } R = 1 \text{ meter} \right)$$

and recalling that the open circuit receiving response M is related to the short circuit transmitting response S through J , we can write

$$\frac{V_{o.c.}}{|p|} = M = JS = \frac{2R\lambda}{\rho_w c_w} \frac{|p|}{I_3}.$$

Replacing k^* by $k^\dagger = k^* [1 + (k_p^2)_{mix}]$ because of electromechanical coupling, we introduce Eq. 27 into this equation and obtain the open circuit receiving response of the sphere:

$$\frac{V_{o.c.}}{|p|} = \frac{\sqrt{2} (k_p)_{hom} J'}{\omega C^s} \sqrt{\frac{\rho_w c_w \bar{c}_{11}^E (1+\nu) [1 + (k_p^2)_{mix}] \epsilon_{33}^S r^s}{t \left[r^{s2} + \left(\omega m^s - \frac{k^\dagger}{\omega} \right)^2 \right]}} \quad (29)$$

Numerical Example: At 14 kc the factor J has a value

$$J = \frac{2\lambda}{\rho_w c_w} = \frac{4\pi}{\omega \rho_w} = \frac{4\pi}{2\pi \times 14 \times 10^3 \times 10^3} \\ = \frac{1}{7} \times 10^{-6}.$$

From the results of the previous computations on a 5-in sphere

$$\frac{V_{o.c.}}{|p|} = \frac{1}{7} \times 10^{-6} \times 5110 = 7.3 \times 10^{-4} \frac{\text{volt-meter}^2}{\text{newton}} = 7.3 \times 10^{-5} \frac{\text{volt}}{\mu\text{bar}} \\ = -82.7 \text{ db ref. } 1 \text{ volt}/\mu\text{bar}.$$

TRANSMITTING EFFICIENCY

The dissipation factor r^* is the useful radiation load delivered through the agency of the applied electrical power. If we now assume an internal mechanical (parasitic) resistance r_m proportional to velocity, we can write the total mechanical dissipation factor as $r^*(1 + \chi)$, where $\chi = r_m/r^*$. The rms power P_{a+m} absorbed by both radiation and mechanical damping is found by replacing r^* by $r^*(1 + \chi)$ in Eq. (24), and taking 1/2 the resultant magnitude since peak voltages are employed. The rms power P_d absorbed by the dielectric resistance of internal damping δ is given by

$$P_d = \frac{V_3^2}{2R_D} = \frac{V_3^2}{2} \omega C^s \tan \delta. \quad (30)$$

Adding the power absorbed mechanically to the power absorbed electrically we obtain the total absorbed power P_T . The efficiency η is therefore

$$\eta = \frac{P'_a}{P'_I} = \frac{P'_a}{P_{a+m} + P_d}$$

where P'_a is computed from Eq. (24) with r^{*2} in the denominator replaced by $r^{*2}(1 + \chi)^2$. Substituting all the derived formulas we obtain

$$\eta = \frac{1}{1 + \chi + \frac{\omega C^s \tan \delta}{\left(\frac{8\pi(k_p^2)_{\text{hom}} \bar{\epsilon}_{11}^E (1 + \nu) [1 + (k_p^2)_{\text{mix}}] \epsilon_{33}^S r^{*2}}{t \left\{ [r^*(1 + \chi)]^2 + \left(\omega m^* - \frac{k^*}{\omega} \right)^2 \right\}} \right)}} \quad (31)$$

Hence at velocity resonance

$$\eta = \frac{1}{1 + \chi + \frac{\rho_w c_w (k_R R_o)^2 \omega C^s \tan \delta (1 + \chi)^2}{8\pi(k_p^2)_{\text{hom}} \bar{\epsilon}_{11}^E (1 + \nu) [1 + (k_p^2)_{\text{mix}}] \epsilon_{33}^S (1 + k_R^2 R_o^2)}} \quad (32)$$

Numerical Example: Continuing the analysis of the 5-in. lead titanate-lead zirconate sphere and allowing $\tan \delta = 0.03$, we find that

$$\eta = \frac{1}{1 + \chi + 0.3(1 + \chi)^2}$$

In the absence of any parasitic mechanical damping the efficiency is 77 percent.

CONCLUSION

We have presented a complete account of the acoustic performance of a radially pulsating piezoceramic sphere. The results are restricted to narrow grounds of validity through the basic assumptions used in the analysis. In particular, the shell is presumed to be "very thin," to move with infinitesimal displacements, to be under the action of weak electric fields, and to have no parasitic losses, except when these are specifically mentioned. The extent to which the performance of an ideal sphere described here differs from that of an actual sphere will depend principally upon the magnitude of electro-mechanical coupling achieved in manufacture. Excessive values of $\tan \delta$ (> 0.03) will also reduce output and must be considered in judging the usefulness of the derived equations.

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Appendix A

DESIGN GRAPHS

A series of design graphs based upon the formulas derived in the main body of the report and on the material constants listed in Table A1 is presented in the following pages. Due to the limitations of planar analysis as applied to a sphere and the uncertainty of the values of material constants in a given (physically real) sphere, it is best to regard the numerical results found from these graphs as convenient guides only, not assuming an accuracy better than ± 7 percent. A few comments on the construction of these graphs will be appropriate here. In the course of these comments we will consider a typical design problem illustrating their use.

TABLE A1
Material Constants of Polarized Ceramics

	Ceramic B (BaTiO ₃ - CaTiO ₃)	PZT-4 (PbTiO ₃ - PbZrO ₃)	PZT-5 (PbTiO ₃ - PbZrO ₃)
\bar{c}_{11}^E (newton/meter ²)	127.9×10^9	91.3×10^9	74.7×10^9
\bar{c}_{12}^E (newton/meter ²)	38.3×10^9	27.9×10^9	23.0×10^9
S_{11}^E (meter ² /newton)	8.62×10^{-12}	12.05×10^{-12}	12.05×10^{-12}
$(k_p)_{hom}$	0.33	0.48	0.54
d_{31} (coulomb/newton)	-58×10^{-12}	-97×10^{-12}	-140×10^{-12}
\bar{e}_{31} (planar) (newton/meter-volt)	-9.45	-10.38	-12.01
ϵ_{33}^S (coulomb/meter-volt)	9.64×10^{-9}	7.89×10^{-9}	10.24×10^{-9}
ϵ_{33}^T (coulomb/meter-volt)	10.7×10^{-9}	9.7×10^{-9}	13.22×10^{-9}
ρ_c (kilogram/meter ³)	5.4×10^3	7.6×10^3	7.6×10^3
$\bar{z}_p = [2\bar{z}_{11}^D(1+\nu)/\rho_c]^{1/2}$ (meter/second)	8.28×10^3	6.2×10^3	5.73×10^3
ν	0.3	0.3	0.3

Figures A1 - Resonant Frequency

The curves in Figs. A1 are numerical values of Eq. (10), except that \bar{z}_{11}^F is used in place of \bar{z}_{11}^D . The resonant frequencies shown are correct as long as the shell is very thin, that is, as long as R_o/t is greater than 10. Let us begin our typical design problem by requiring a PZT-4 sphere (see Table A1) to vibrate resonantly in water at 1 kc. From Fig. A1 we find that R_o is 35 inches in air. The final design radius will be smaller because of water loading.

Figure A2 - Mechanical Q

Figure A2 holds for both materials discussed in the report. It is based upon Eq. (16b), for which the condition $k^2 R_o^2 \gg 1$ holds. In our problem we specify a mechanical Q of 2. From Fig. A2 we see that R_o/t must be about 15, and hence that the thickness t must be 2-1/3 inches.

Figures A3 - Radial Displacement

Figures A3 are based upon Eq. (11). The velocity of sound in water is taken at $c_w = 1.55 \times 10^3$ meter/sec. For a value of $R_o/t = 15$, we judge the radial displacement to be 2.75×10^{-9} meter/volt at $\omega = \omega_n$, and somewhat higher, say 2.90×10^{-9} meter/sec at true water resonance. The surface velocity per volt is therefore

$$\omega \times \text{displacement} = 2\pi \times 10^3 \times 2.75 \times 10^{-9} = 1.73 \times 10^{-5} \text{ m/volt-sec.}$$

Figure A4 - Surface Area

Figure A4 gives the surface area of spheres as a function of diameter. The surface area of a 70-inch-diameter sphere is 99,000 cm^2 .

Figures A5 - Power Density

The curves in Figs. A5 are numerical plots of Eq. (26). If we allow a surface power density of 1 watt/ cm^2 , we will need approximately 6900 volts (peak) electrical drive. Since the thickness is 2-1/3 inches, or 8.45 cm, the maximum electrical gradient will be 818 volts/cm. The material of the sphere is evidently far from its power absorption potential, since a value of 2400 peak volts/cm could be used with impunity. The total power for 100-percent efficiency and for the coupling cited will be about 100,000 watts.

Figures A6 - Power Curves

Figures A6 are based upon Eq. (24). We have plotted, however, the rms values (1/2 of the formula) instead of the peak values. In our design problem, let us require the power (for 100-percent efficiency) at 1/2 the design frequency, that is, at $\omega/\omega_n = 0.5$. We find, from Fig. A6b, that the real power output is approximately 250/2285 or 0.09 of its value at resonance. Hence, the power at 500 cps will be about 11 kw.

Figures A7 - Resonant Pressure Amplitude

Figures A7 are based on Eqs. (28a) and (28b). From them we find that for a drive of 6900 peak volts (76.8 db ref. 1 volt) the source level at 1 meter will be 125 db ref. 1 μbar (Fig. A7A) and the transmitting response will be +89.2 db ref. 1 μbar per ampere (Fig. A7b).

Figure A8 - Receiving Response

The curves in Fig. A8 are based on Eq. (29), computed at velocity resonance. For a thickness of 2-1/3 inches the resonant receiving response will be -65.0 db ref. 1 volt/ μbar .

Figure A9 - Blocked Reactance at Resonance (X_B)

The curves in Fig. A9 are computed from the simplified assumption of $X_B = 1/\omega_R C^*$. At velocity resonance $\omega_R C^*$ is a function of t/R_0 only. For $t/R_0 = 1/15 = 0.067$, $X_B = 120$ ohms.

Figure A10 - Electrical Q_E Curves

For a given material the coupling factor k_p is known. Using the simplified relation that $Q_E Q_M = [1 - (k_p^2)_{\text{hom}}]/(k_p^2)_{\text{hom}}$, the curves for the two materials in question were computed as shown in Fig. A10. In the illustrative example $Q_M = 2$; hence $Q_E = 1.7$. Now assuming that Q_E is roughly X_B/R , we compute $R \approx 70$ ohms. The electrical impedance of the sphere is therefore approximately $70 - j 120$ at velocity resonance.

Acknowledgement

The author is indebted to Mrs. Berthel K. Carmichael for part of the computation work involved in constructing these charts.

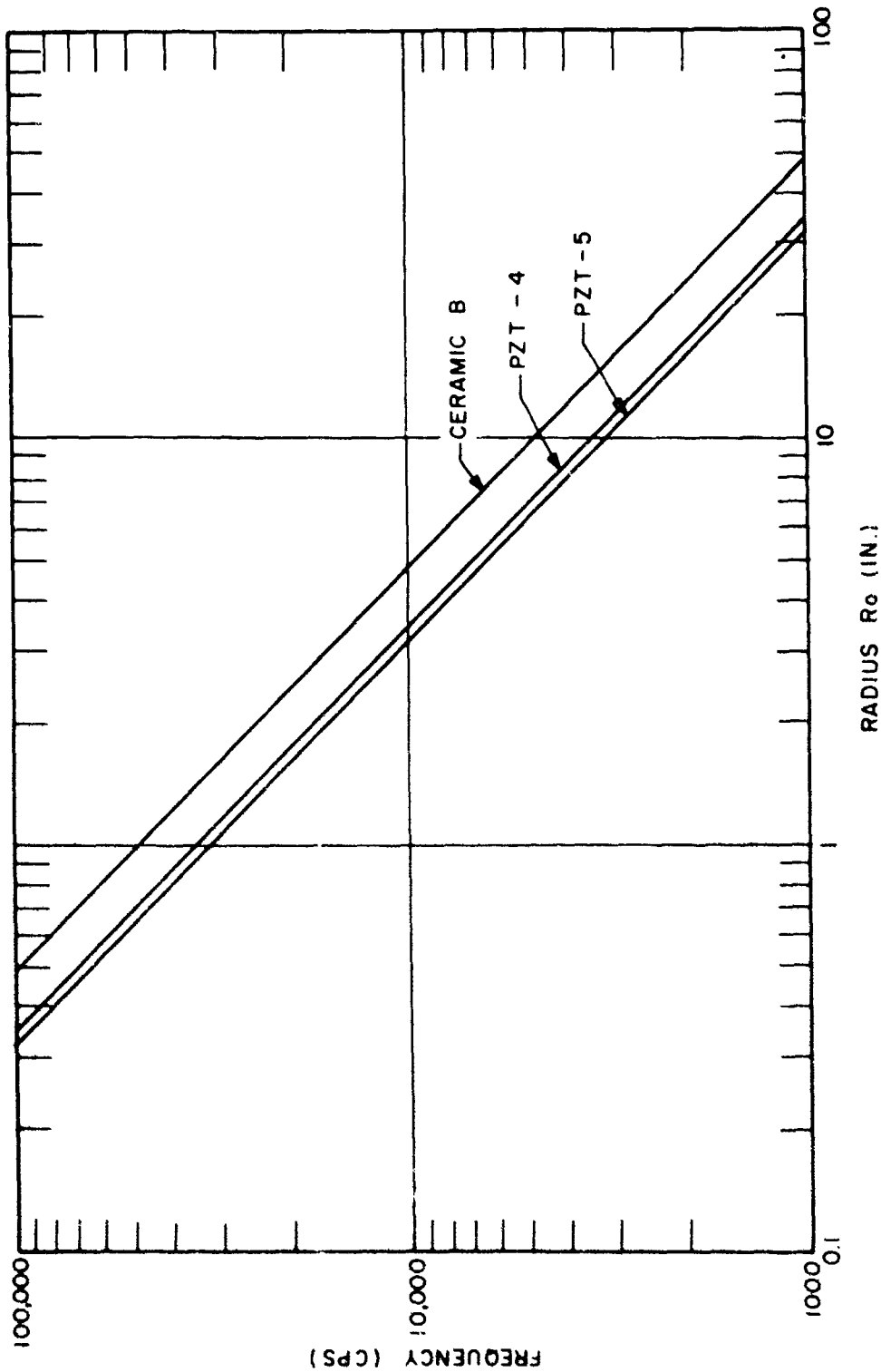


Figure Ala - Resonant frequency in air of thin ceramic spheres versus the radius (R_0) of the spheres

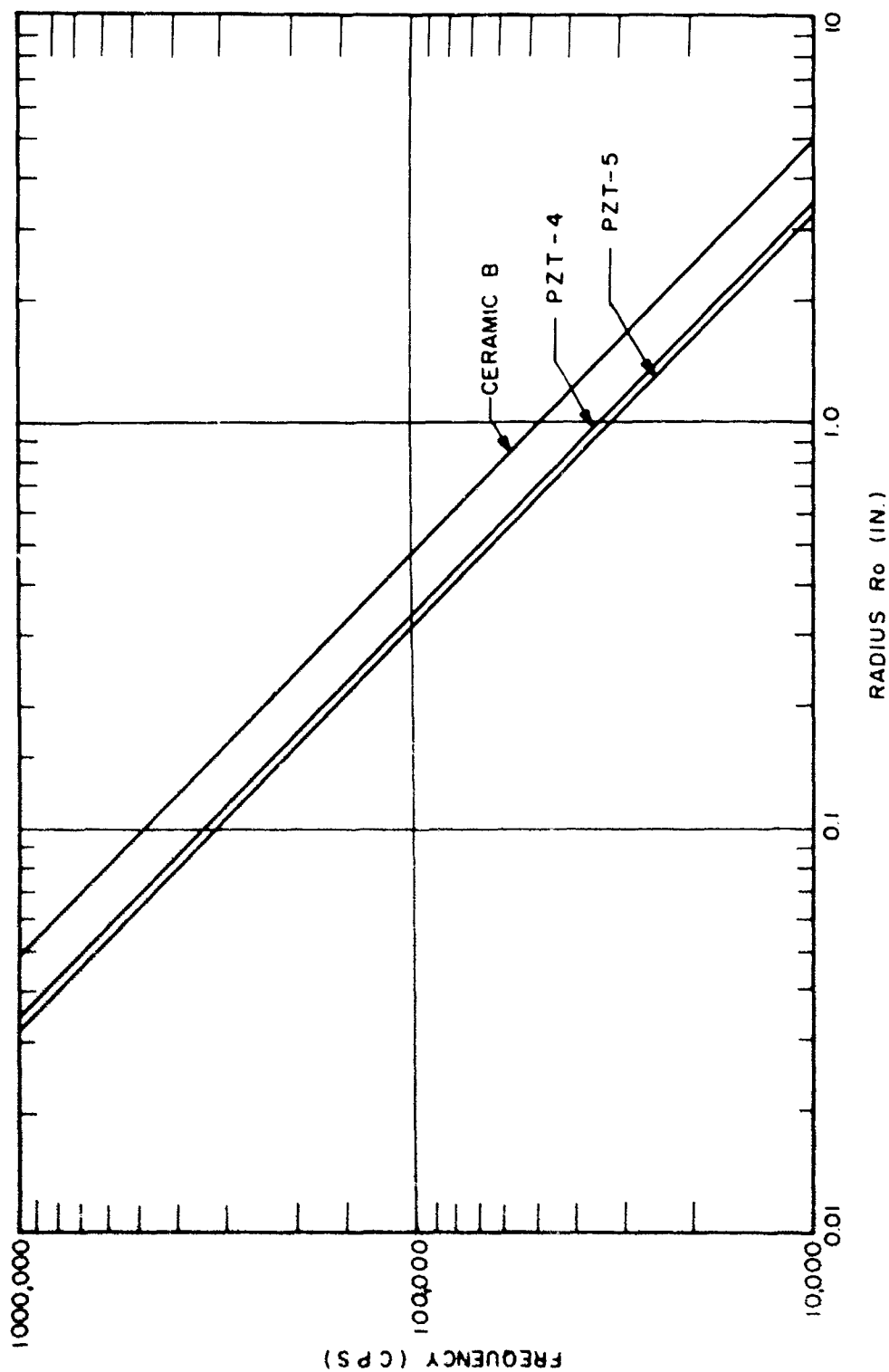


Figure A1b - Resonant frequency in air of thin ceramic spheres versus the radius (R_0) of the spheres

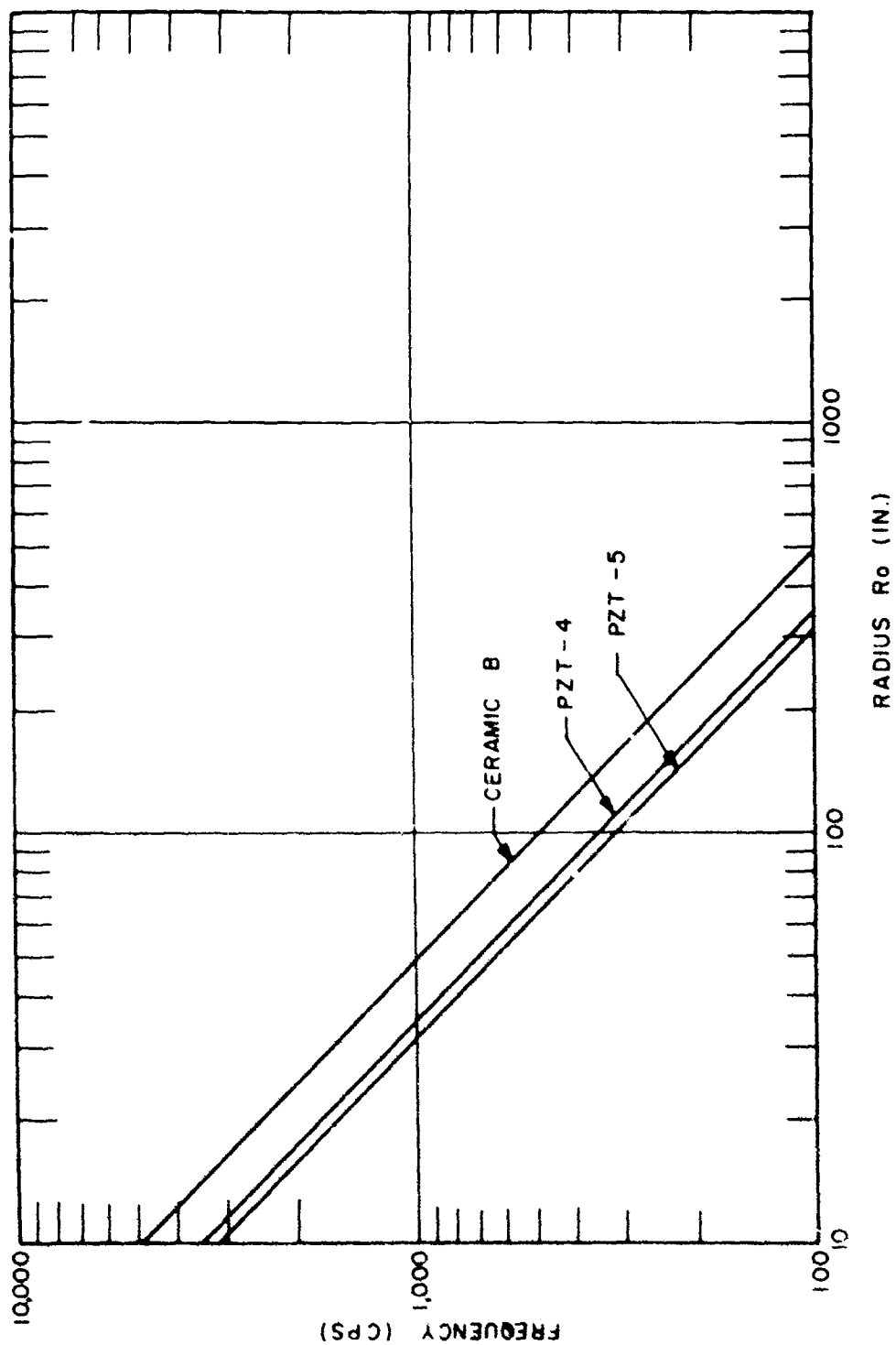


Figure A1c - Resonant frequency in air of thin ceramic spheres versus the radius (R_0) of the spheres

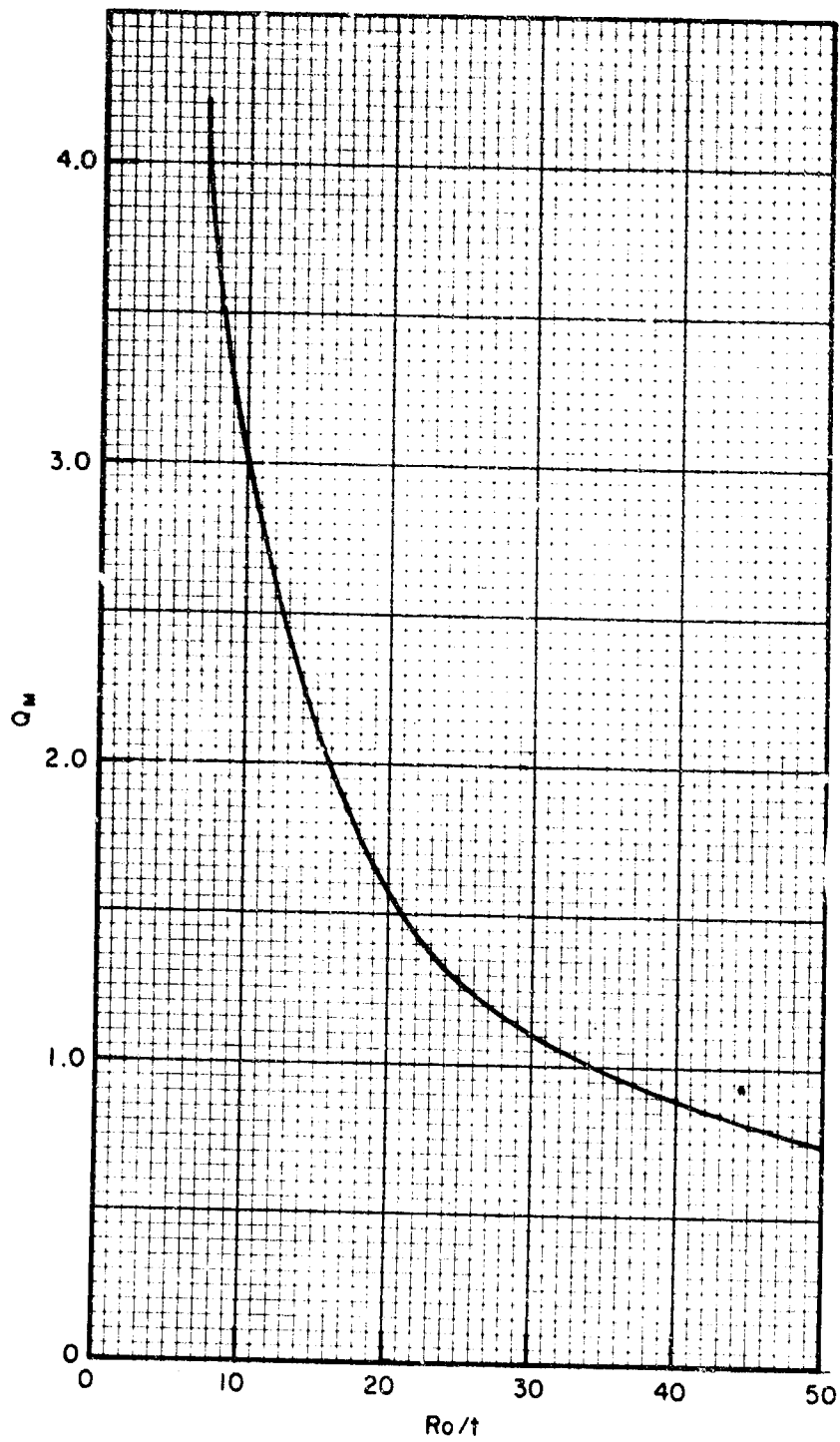


Figure A2 - Approximate mechanical Q of water loaded piezoelectric ceramic spheres versus the radius-to-thickness ratio (R_o/t). This curve applies to all three materials.

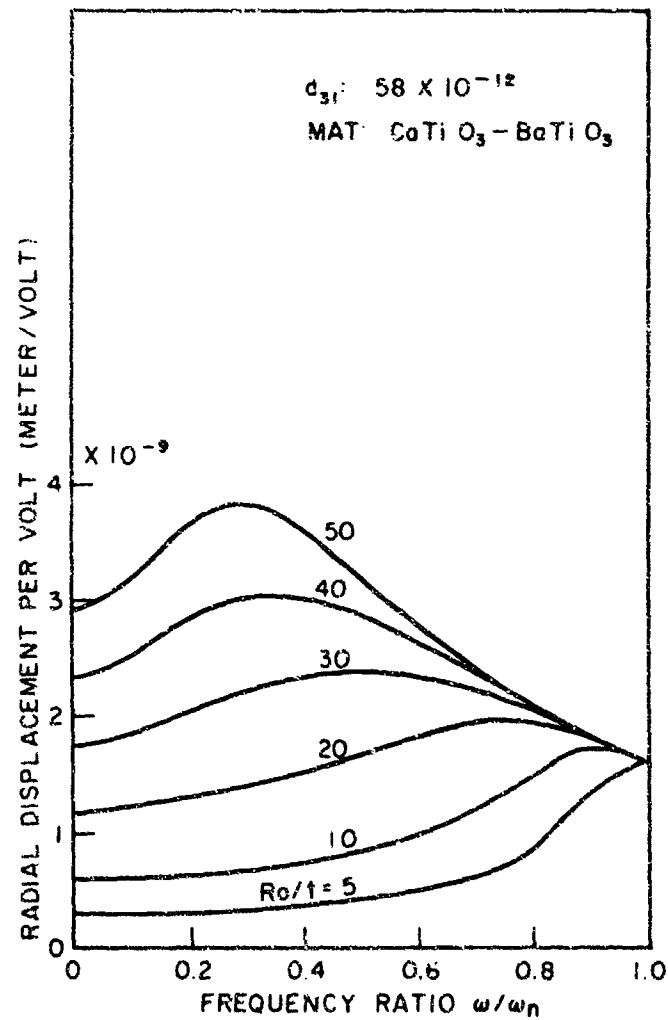


Figure A3a - Radial displacement per volt versus frequency ratio ω/ω_n and the radius-to-thickness ratio of piezoelectric ceramic spheres

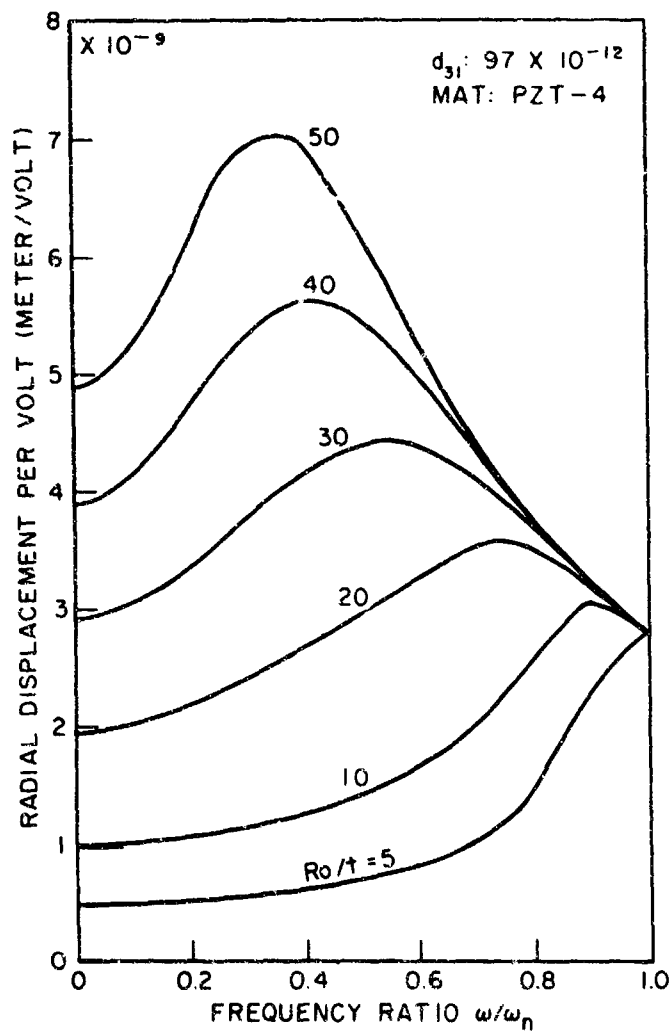


Figure A3b - Radial displacement per volt versus frequency ratio ω/ω_n and the radius-to-thickness ratio of piezoelectric ceramic spheres

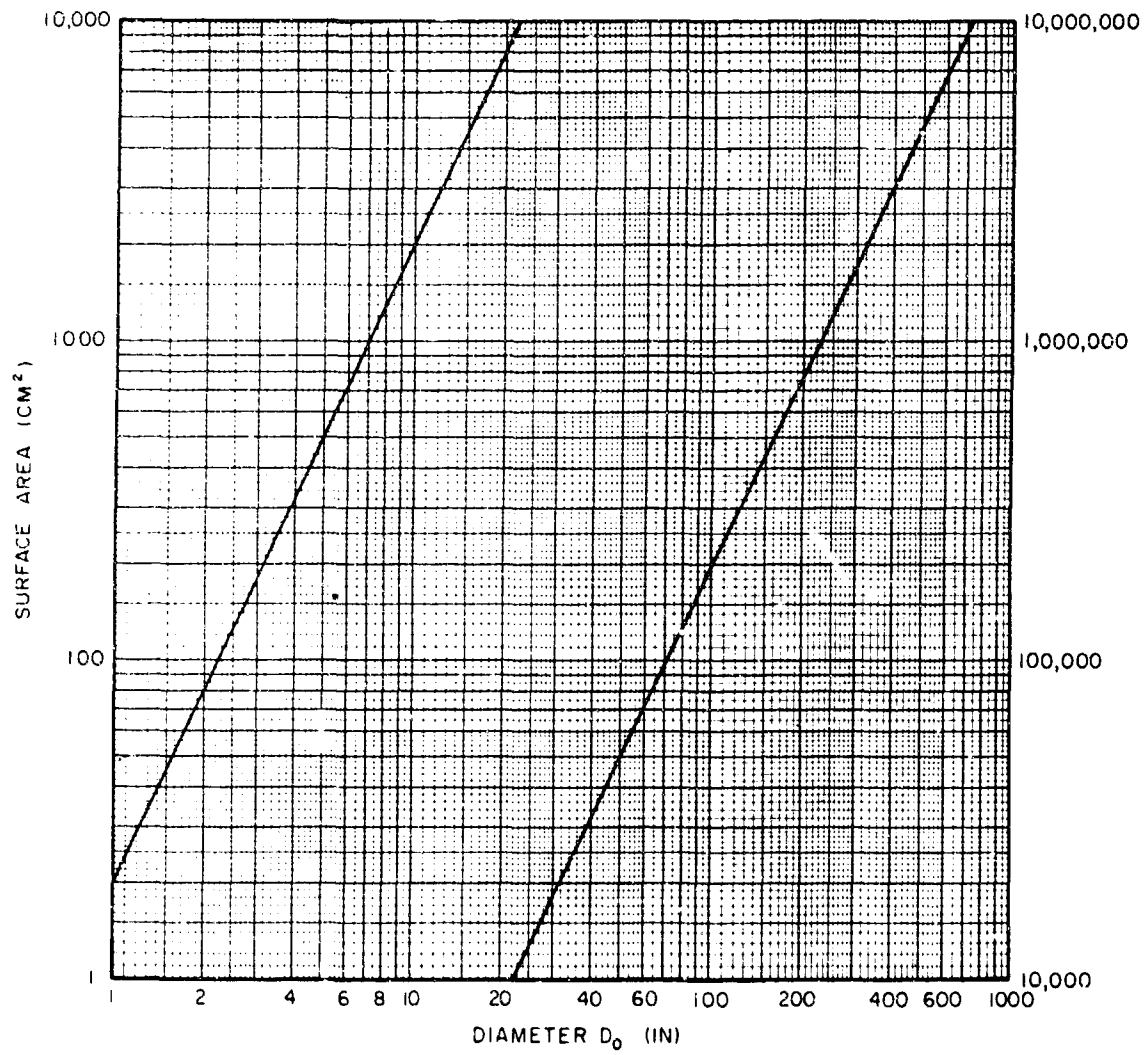


Figure A4 - Surface area of a sphere versus diameter

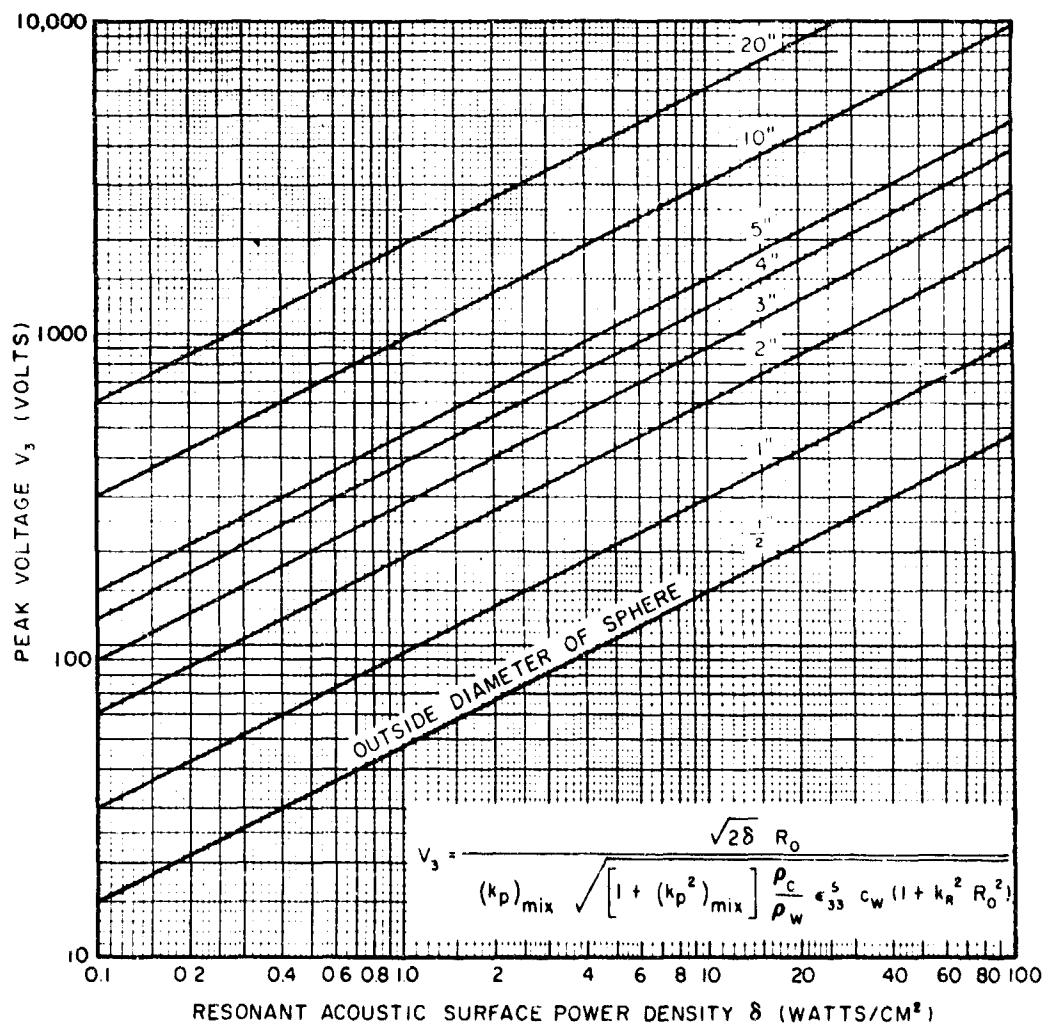


Figure A5a - Peak voltage required (100% efficiency) versus the rms value of δ , the desired acoustic surface power density. The values shown are for PZT-4 and PZT-5; Ceramic B values can be obtained by multiplying V_3 by 1.165.

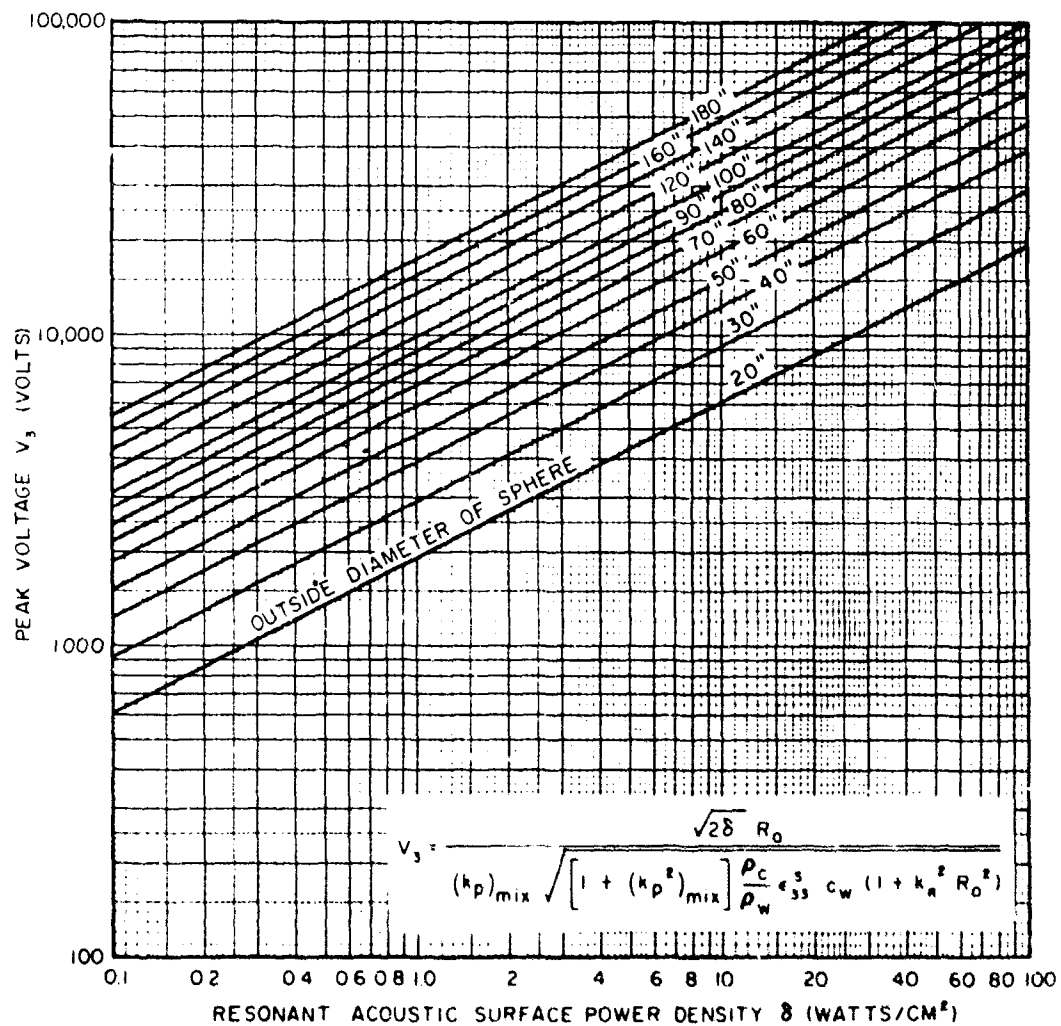


Figure A5b - Peak voltage required (100% efficiency) versus the rms value of δ , the desired acoustic surface power density. The values shown are for PZT-4 and PZT-5; Ceramic B values can be obtained by multiplying V_3 by 1.165.

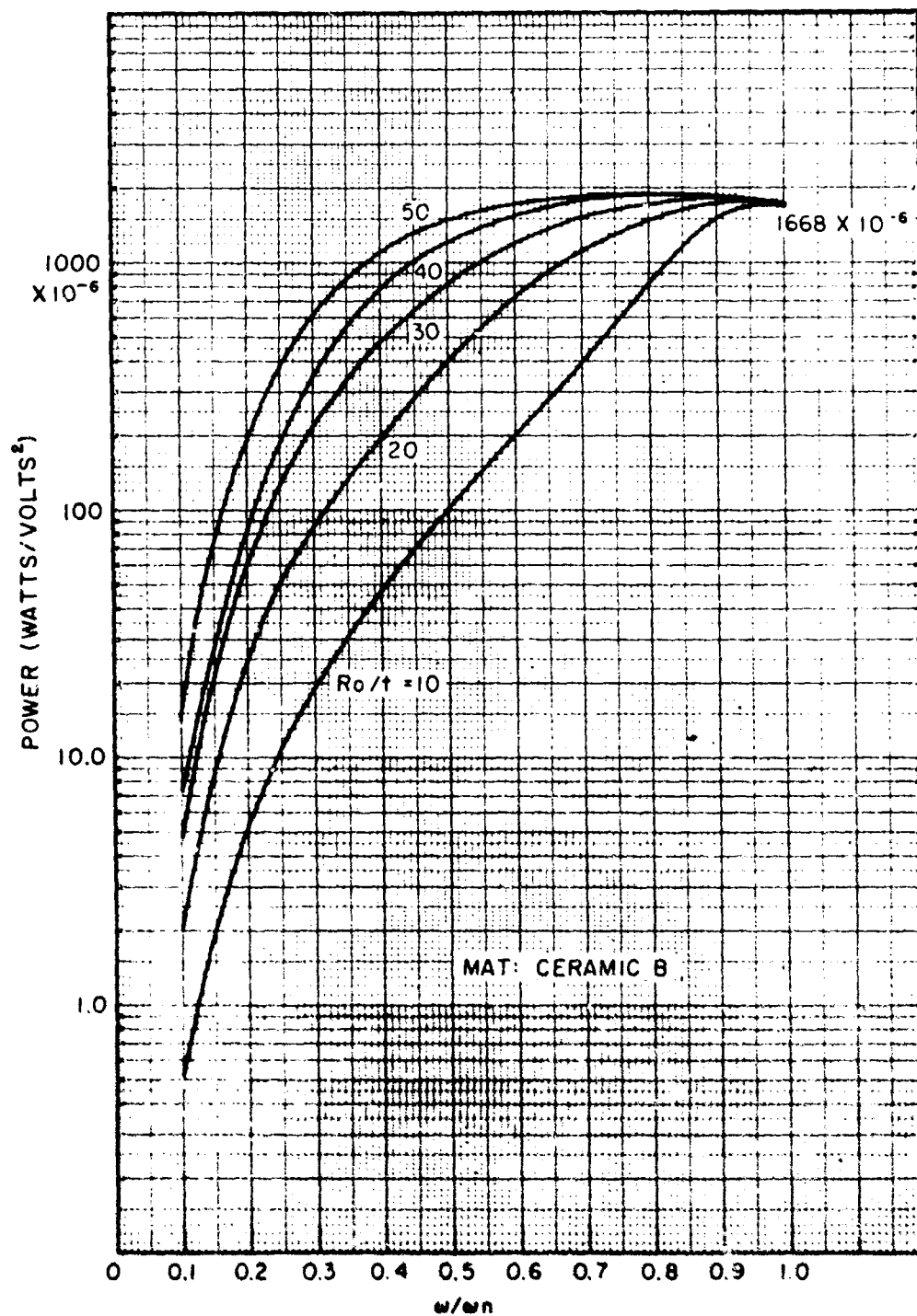


Figure A6a - Real acoustic power (rms watts) per squared peak applied voltage versus frequency ratio ω/ω_n for piezoceramic spheres with various R_0/t values

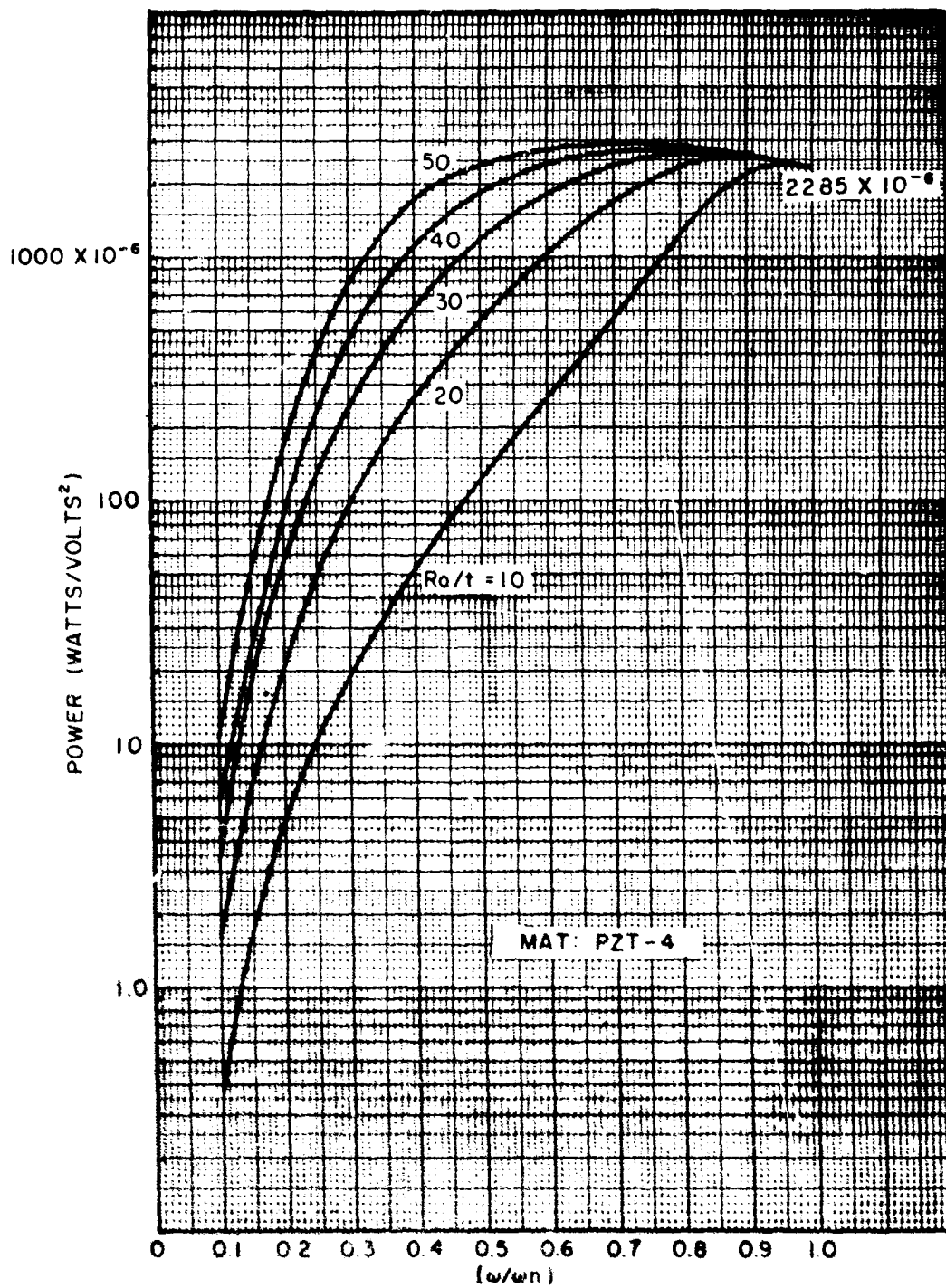


Figure A6b - Real acoustic power (rms watts) per squared peak applied voltage versus frequency ratio ω/ω_n for piezoceramic spheres with various R_0/t values

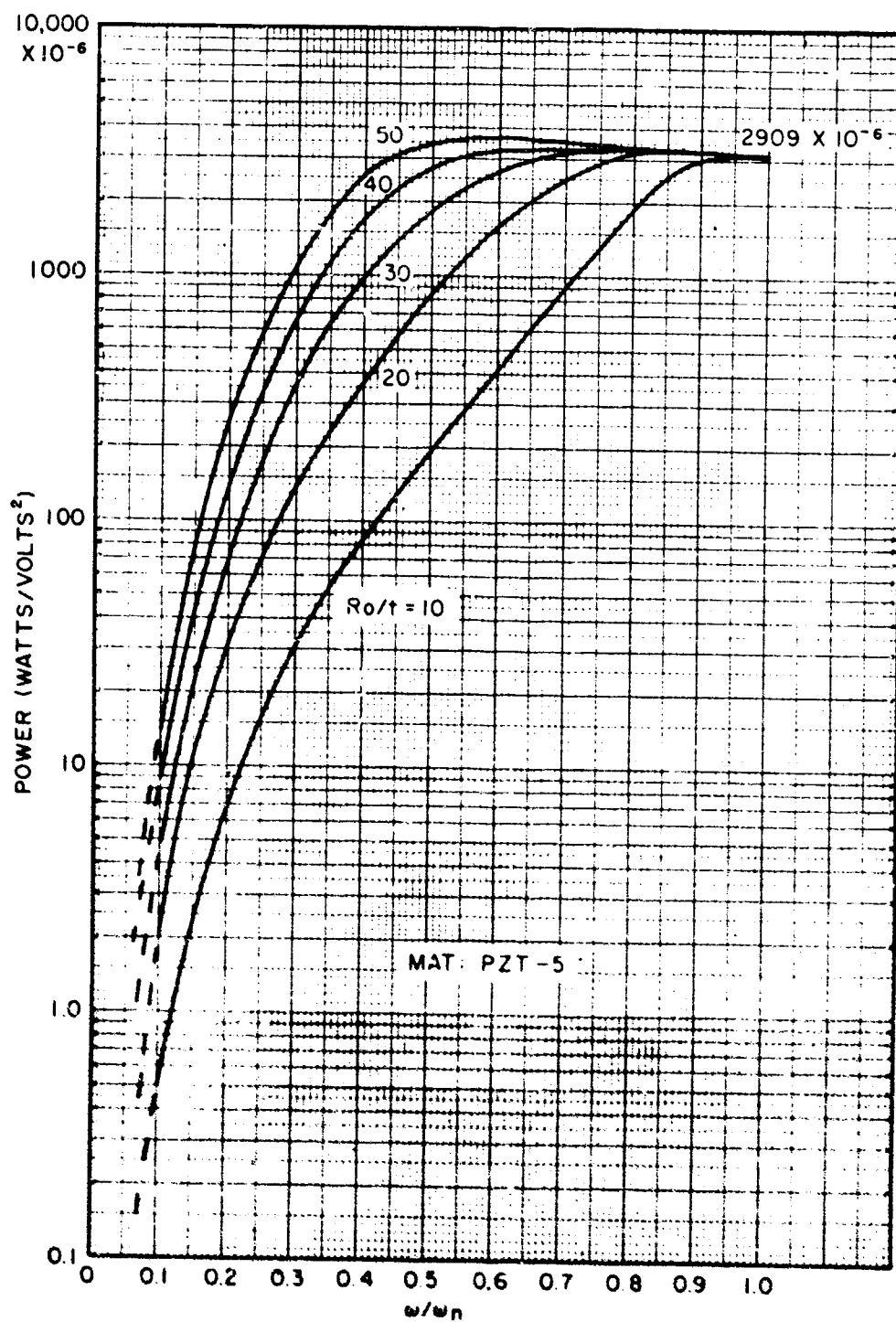


Figure A6c - Real acoustic power (rms watts) per squared peak applied voltage versus frequency ratio ω/ω_n for piezoceramic spheres with various R_0/t values

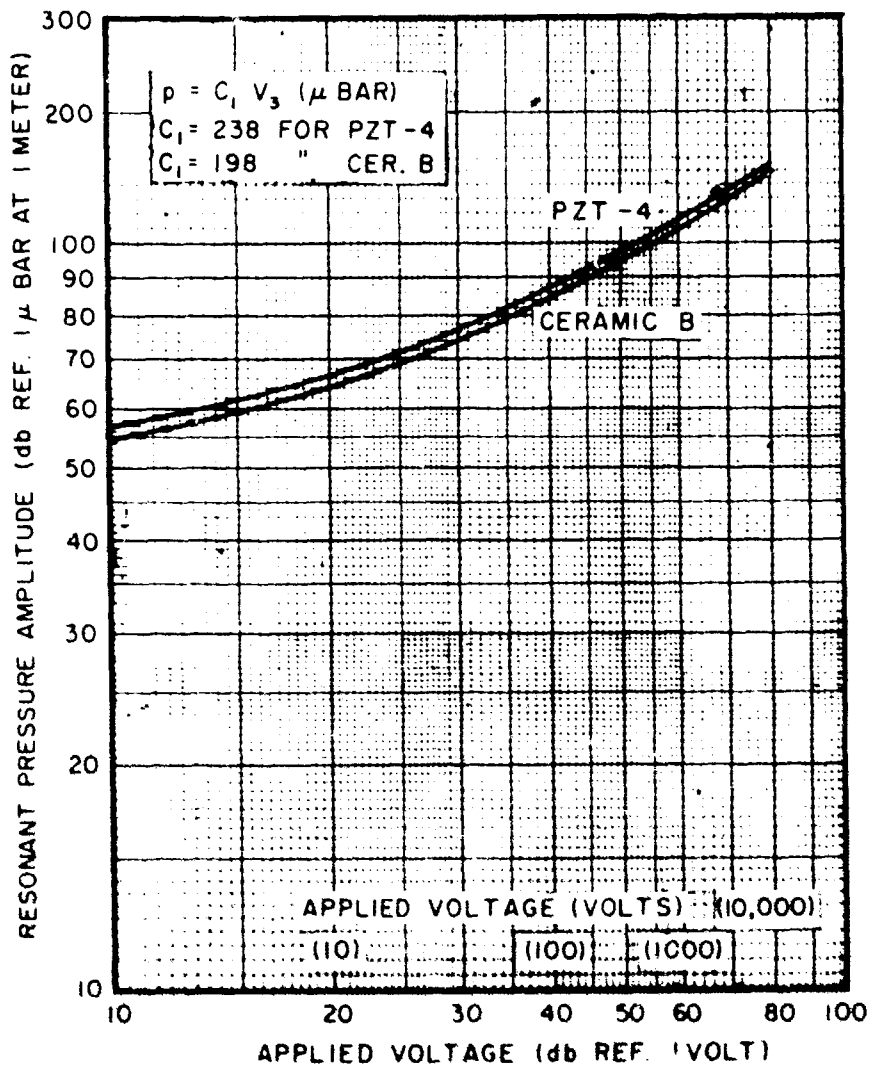


Figure A7a - Resonant pressure amplitude versus peak applied voltage (100% efficiency)

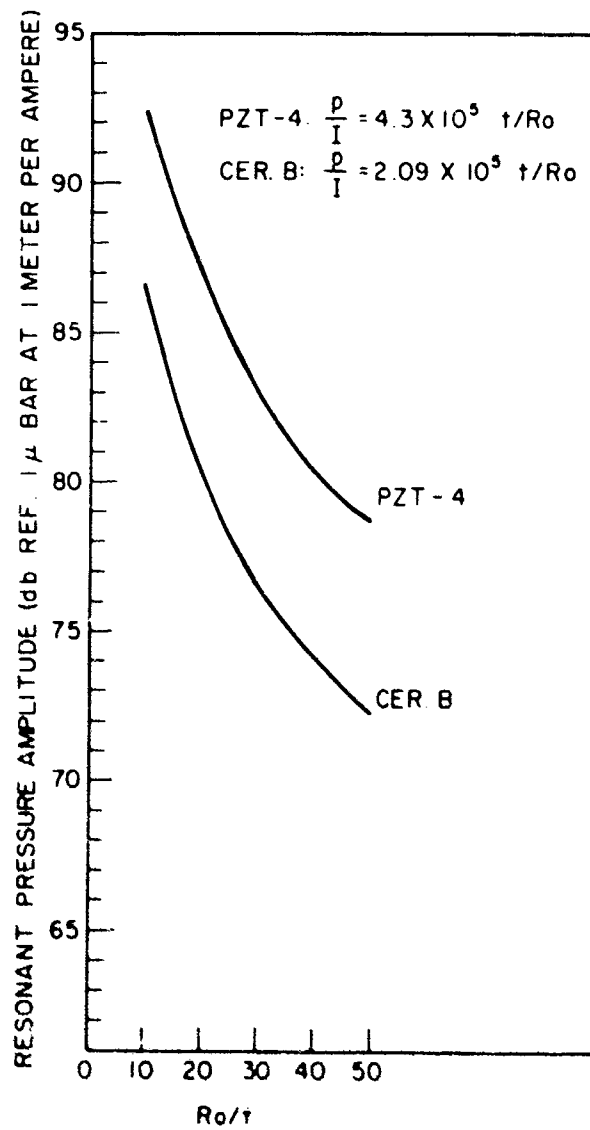


Figure A7b - Resonant pressure amplitude versus the radius-to-thickness ratio

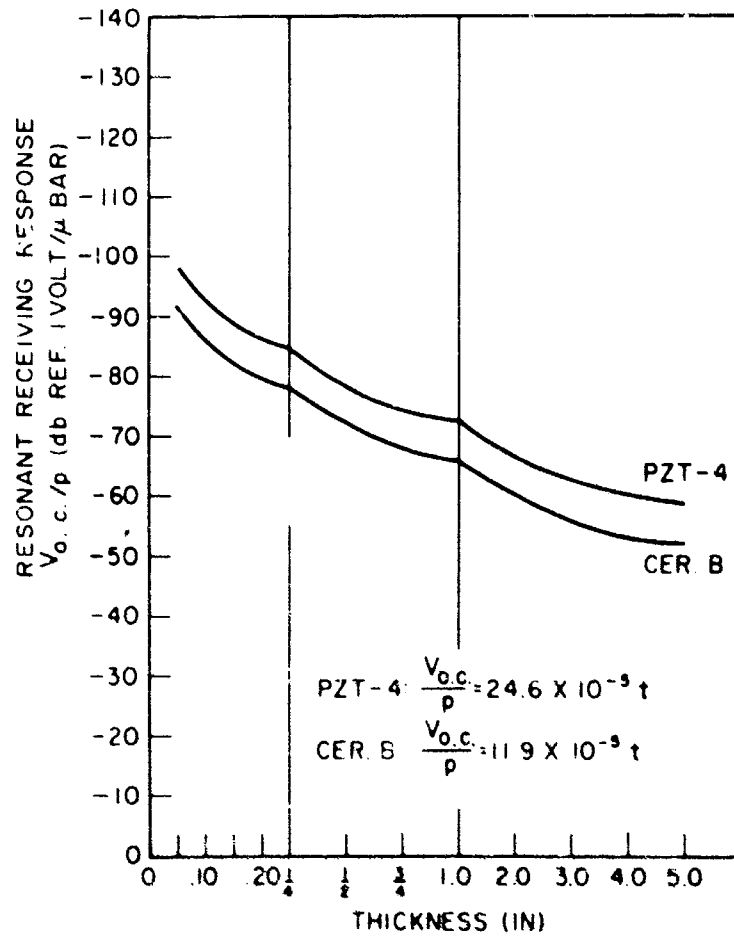


Figure A8 - Resonant open circuit receiving response versus shell thickness t

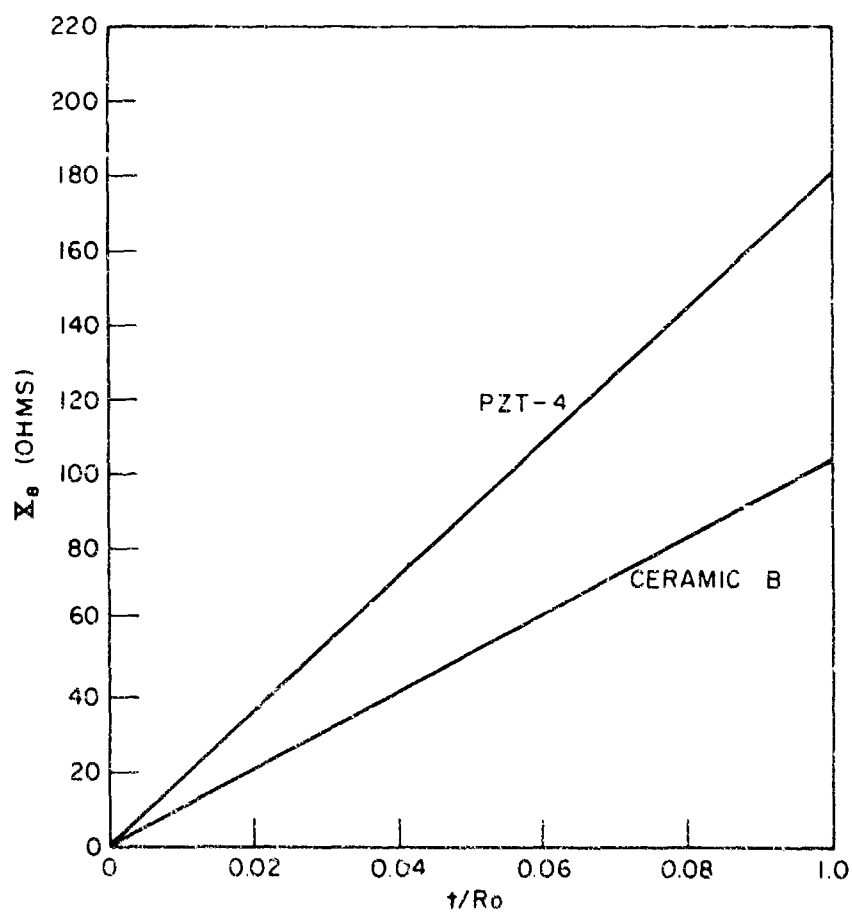


Figure A9 - Blocked reactive impedance (X_B) at velocity resonance versus the thickness-to-radius ratio

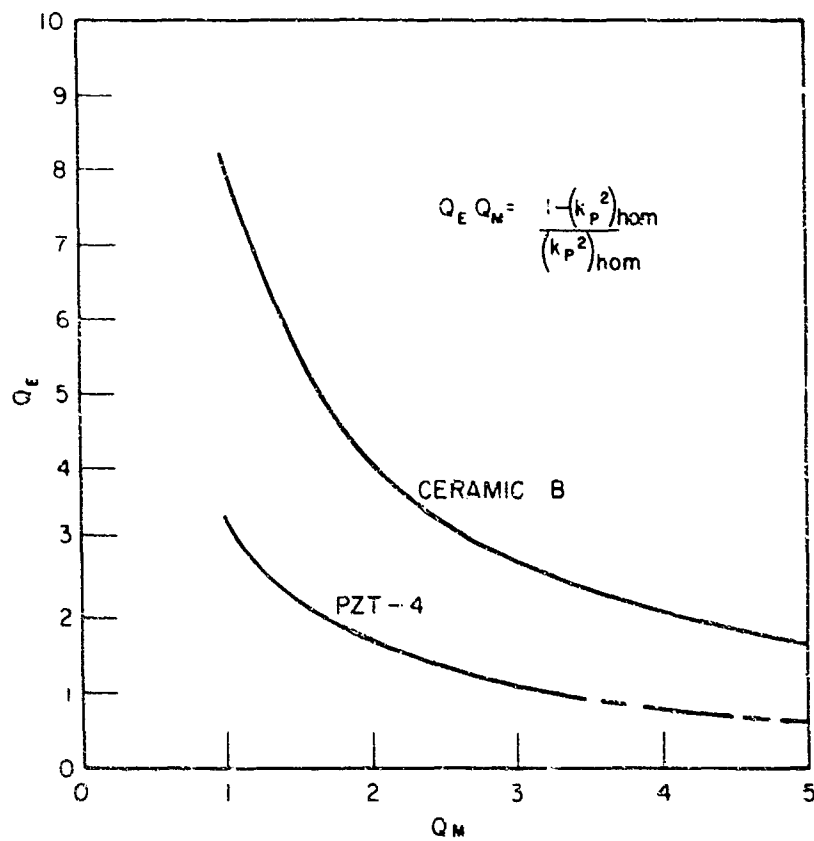


Figure A10 - Electrical Q_E versus mechanical Q_M of piezoceramic spheres

LIST OF SYMBOLS

$\bar{c}_{11}, \bar{c}_{12}$	Planar stiffness coefficients (newton/meter ²)
C^*	Capacity at zero strain (farad)
c_w	Compressional velocity of sound in water (meter/second)
d_{31}	Piezo modulus
\bar{D}_3	Electric displacement (coulomb/meter ²)
D_o	Outside diameter of sphere (meter)
D, E	Superscripts indicating at constant electric displacement and constant field respectively
E_3	Electric field (volt/meter)
\bar{e}_{31}	Planar piezo modulus (newton/meter-volt)
F_r	Body force per unit of mass
F_o	$2\bar{e}_{31}\bar{D}_3/\epsilon_{33}R_o$
I_3	Applied current (ampere)
J	Reciprocity factor (meter ⁵ /newton-second)
k	Wave number (meter ⁻¹)
k^*	Defined by Eq. (12d)
k_n	Wave number at velocity resonance in air (meter ⁻¹)
$(k_p)_{mix}$	Mixed coefficient of electromechanical coupling
$(k_p)_{hom}$	Homogeneous coefficient of electromechanical coupling
k_R	Wave number at velocity resonance in water (meter ⁻¹)
m^2	Defined by Eq. (12b)
p	Alternating acoustic pressure (newton/meter ²)
P_a	Acoustic power (watt)
Q_m^*	Quality Factor (= mechanical Q_m at velocity resonance)

r	Radial coordinate
r^*	Defined by Eq. (12c)
r_m	Internal mechanical resistance
r_f	Distance in far field (meter)
R_o	Outside radius of sphere (meter)
s	Superscript
S_1, S_2	Planar strains (meter/meter)
s_{11}^E	Compliance (meter ² /newton)
t	Shell thickness (meter)
T_1, T_2	Planar stresses (newton/meter ²)
T_{33}, T_{31}, T_{32}	Stresses (newton/meter ²)
v	Radial velocity in steady state (meter/second)
V_3	Applied voltage (volt)
$V_{o.c.}$	Open circuit voltage (volt)
α_d	Temporal damping constant (Eq. 19)
ϵ_{33}^S	Dielectric constant at zero strain (coulomb/meter-volt)
ϵ_{33}^T	Dielectric constant at zero stress (coulomb/meter-volt)
θ	Spherical coordinate
ν	Poisson's ratio
ξ	Radial displacement (meter)
$\dot{\xi}$	Radial velocity (meter/second)
ρ_c	Density of ceramic (newton-second ² /meter ⁴)
ρ_w	Density of water (newton-second ² /meter ⁴)
ϕ	Spherical coordinate and phase angle
χ	r_m/r^*
ω	Frequency (radian/second)
ω_n	Resonant frequency in air (radian/second)
ω_R	Resonant frequency in water (radian/second)